

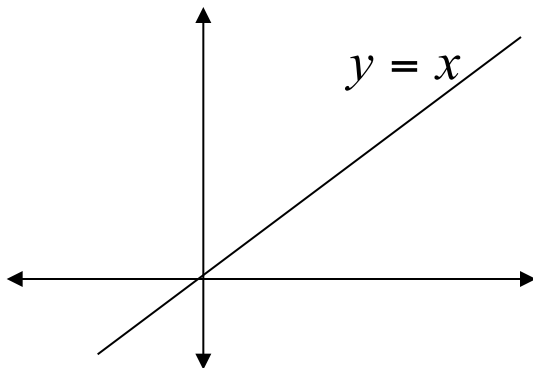
8. FUNGSI TRANSENDEN

8.1 Invers Fungsi

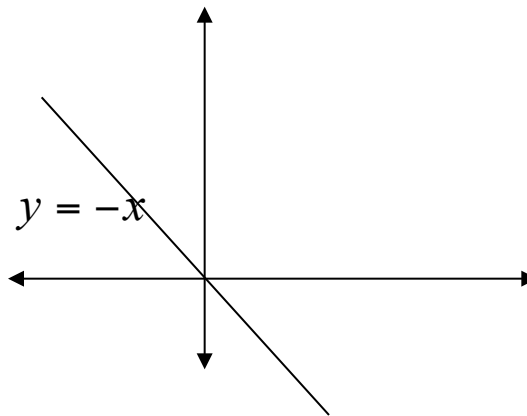
Misalkan $f : D_f \rightarrow R_f$ dengan

$$x \mapsto y = f(x)$$

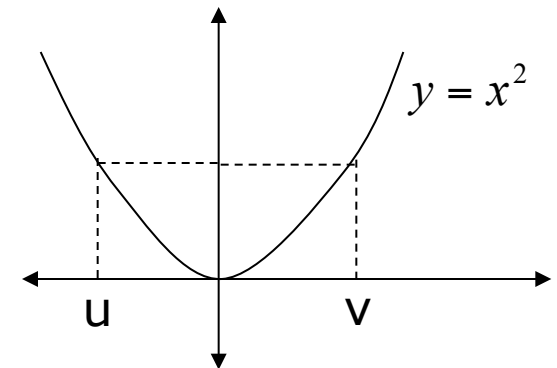
Definisi 8.1 Fungsi $y = f(x)$ disebut satu-satu jika $f(u) = f(v)$ maka $u = v$ atau jika $u \neq v$ maka $f(u) \neq f(v)$



fungsi $y=x$ satu-satu



fungsi $y=-x$ satu-satu



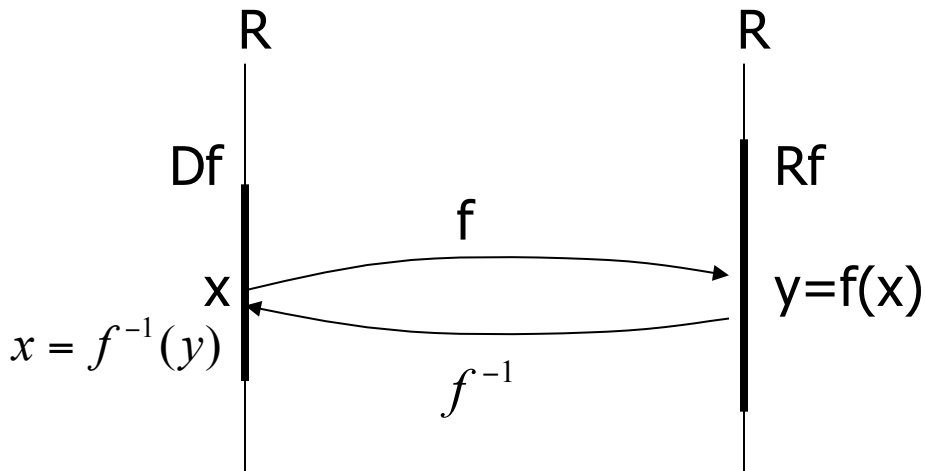
fungsi $y = x^2$ tidak satu-satu

Secara geometri grafik fungsi satu-satu dan garis yang sejajar dengan sumbu x berpotongan di satu titik.

Teorema : Jika fungsi f satu-satu maka f mempunyai invers
notasi f^{-1}

$$f^{-1} : R_f \rightarrow D_f$$

$$y \mapsto x = f^{-1}(y)$$



Berlaku hubungan

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

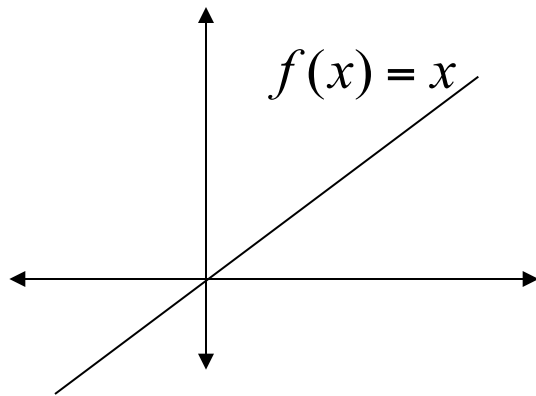
$$D_{f^{-1}} = R_f, \quad R_{f^{-1}} = D_f$$

Teorema : jika f monoton murni(selalu naik/selalu turun) maka f mempunyai invers

$$f(x)=x$$

$$f'(x) = 1 > 0, \forall x \in R$$

f selalu naik

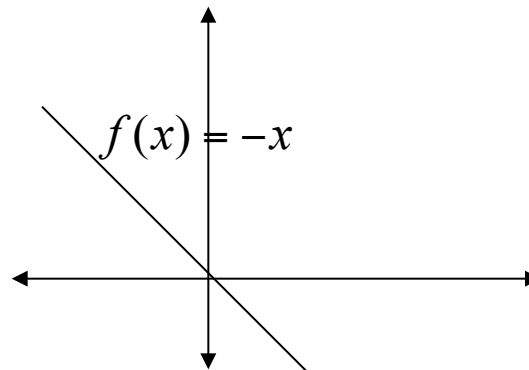


f^{-1} ada

$$f(x)=-x$$

$$f'(x) = -1 < 0, \forall x \in R$$

f selalu turun

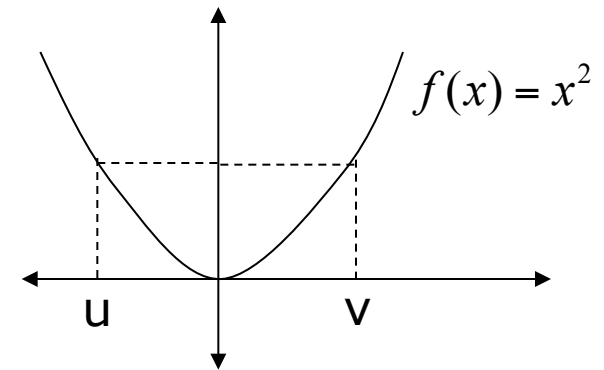


f^{-1} ada

$$f'(x) = 2x = \begin{cases} > 0, x > 0 \\ < 0, x < 0 \end{cases}$$

f naik untuk $x > 0$

turun untuk $x < 0$



f^{-1} tidak ada

Contoh : Diketahui $f(x) = \frac{x-1}{x+2}$

- Periksa apakah f mempunyai invers
- Jika ada, tentukan inversnya

Jawab

a. $f'(x) = \frac{1 \cdot (x+2) - 1 \cdot (x-1)}{(x+2)^2} = \frac{3}{(x+2)^2} > 0, \forall x \in Df$

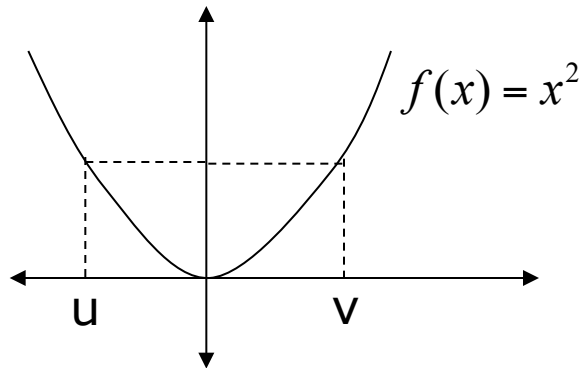
Karena f selalu naik(monoton murni) maka f mempunyai invers

b. Misal $y = \frac{x-1}{x+2}$

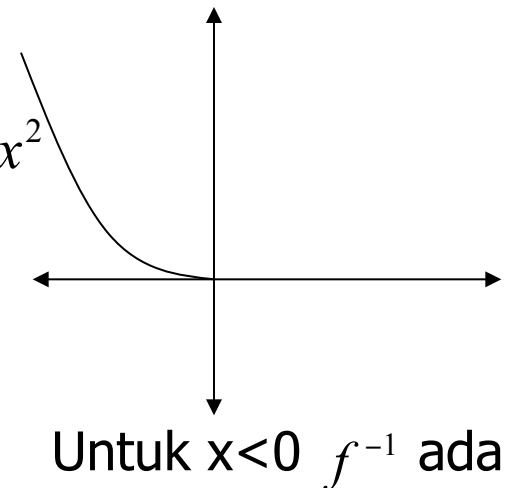
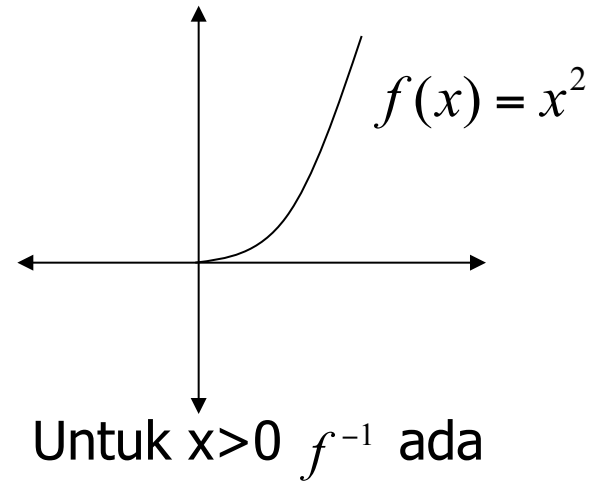
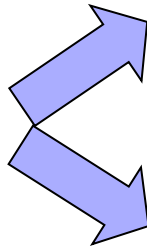
$$xy + 2y = x - 1 \iff xy - x = -2y - 1 \implies x = \frac{-2y - 1}{y - 1}$$

$$f^{-1}(y) = \frac{-2y - 1}{y - 1} \implies f^{-1}(x) = \frac{-2x - 1}{x - 1}$$

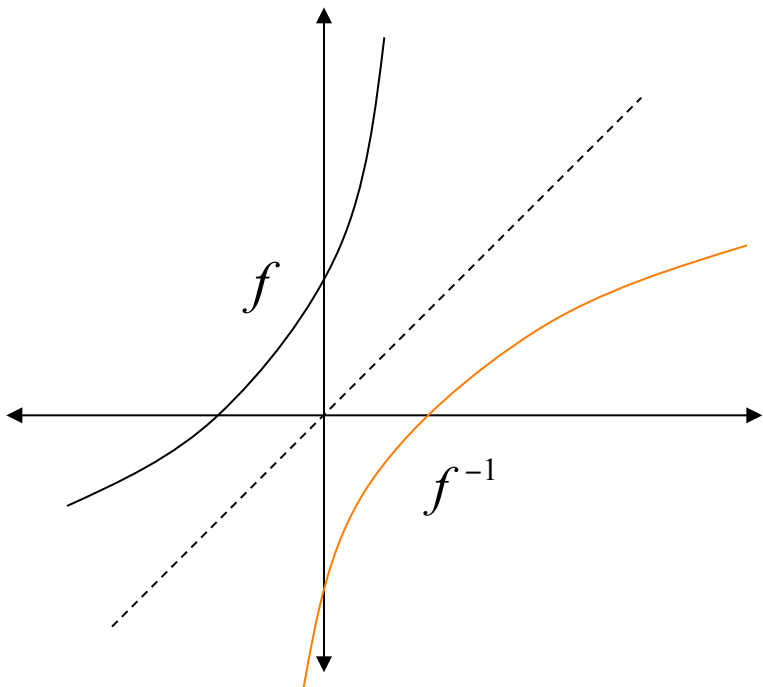
Suatu fungsi yang tidak mempunyai invers pada daerah asalnya dapat dibuat mempunyai invers dengan cara membatasi daerah asalnya.



Untuk $x \in R$ f^{-1} tidak ada



Grafik fungsi invers



Titik (x,y) terletak pada grafik f \iff Titik (y,x) terletak pada grafik f^{-1}

Titik (x,y) dan (y,x) simetri terhadap garis $y=x$

Grafik f dan f^{-1} semetri terhadap garis $y=x$

Turunan fungsi invers

Teorema Misalkan fungsi f monoton murni dan mempunyai turunan pada selang I . Jika $f'(x) \neq 0, x \in I$ maka f^{-1} dapat diturunkan di $y=f(x)$ dan

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Bentuk diatas dapat juga dituliskan sebagai

$$\frac{dx}{dy} = \frac{1}{dy / dx}$$

Contoh Diketahui $f(x) = x^5 + 2x + 1$ tentukan $(f^{-1})'(4)$

Jawab :

$$f'(x) = 5x^4 + 2, y=4 \text{ jika hanya jika } x=1$$

$$\rightarrow (f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{7}$$

Soal Latihan

Tentukan fungsi invers (bila ada) dari

1. $f(x) = x + \frac{1}{x}$, $x > 0$

2. $f(x) = \sqrt[3]{2x - 1}$

3. $f(x) = \sqrt[5]{4x + 2}$

4. $f(x) = \frac{5}{x^2 + 1}$, $x \geq 0$

5. $f(x) = \frac{x + 1}{x - 1}$

6. $f(x) = \frac{2x - 3}{x + 2}$

8.2 Fungsi Logaritma Asli

- Fungsi Logaritma asli (\ln) didefinisikan sebagai :

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

- Dengan Teorema Dasar Kalkulus II, diperoleh :

$$D_x [\ln x] = D_x \left(\int_1^x \frac{1}{t} dt \right) = \frac{1}{x}$$

- Secara umum, jika $u = u(x)$ maka

$$D_x [\ln u] = D_x \left(\int_1^{u(x)} \frac{1}{t} dt \right) = \frac{1}{u} \frac{du}{dx}$$

Contoh : Diberikan $f(x) = \ln(\sin(4x + 2))$

$$\text{maka } f'(x) = \frac{1}{\sin(4x + 2)} D_x(\sin(4x + 2)) = 4 \cot(4x + 2)$$

Jika $y = \ln |x|, x \neq 0$

$$= \begin{cases} \ln x, x > 0 & \longrightarrow y = \ln x \rightarrow y' = \frac{1}{x} \\ \ln(-x), x < 0 & \longrightarrow y = \ln(-x) \rightarrow y' = \frac{-1}{-x} = \frac{1}{x} \end{cases}$$

Jadi, $\frac{d}{dx}(\ln |x|) = \frac{1}{x}, x \neq 0.$

Dari sini diperoleh : $\int \frac{1}{x} dx = \ln |x| + C$

Sifat-sifat Ln :

1. $\ln 1 = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a/b) = \ln(a) - \ln(b)$
4. $\ln a^r = r \ln a$

Contoh: Hitung $\int_0^4 \frac{x^2}{x^3 + 2} dx$

jawab

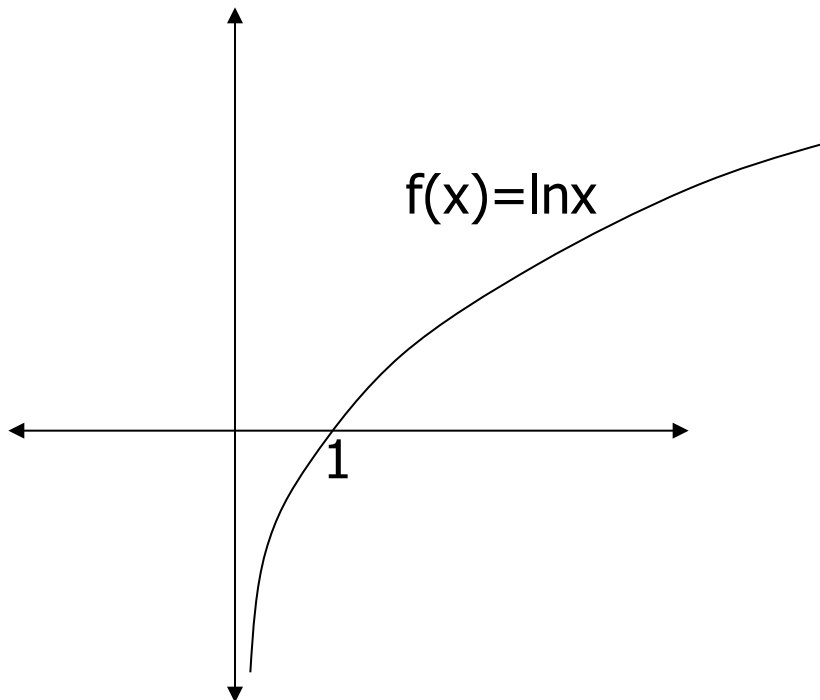
Misal $u = x^3 + 2 \rightarrow du = 3x^2 dx$

$$\begin{aligned}\int \frac{x^2}{x^3 + 2} dx &= \int \frac{x^2}{u} \frac{du}{3x^2} = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |u| + c \\ &= \frac{1}{3} \ln |x^3 + 2| + c\end{aligned}$$

sehingga

$$\int_0^4 \frac{x^2}{x^3 + 2} dx = \frac{1}{3} \ln |x^3 + 2| \Big|_0^4 = \frac{1}{3} (\ln 66 - \ln 2) = \frac{1}{3} \ln 33.$$

Grafik fungsi logaritma asli



Diketahui

a. $f'(x) = \ln x = \int_1^x \frac{dt}{t}, x > 0$

b. $f'(x) = \frac{1}{x} > 0 \forall x \in D_f$

f selalu monoton naik pada D_f

c. $f''(x) = -\frac{1}{x^2} < 0 \forall x \in D_f$

Grafik selalu cekung kebawah

d. $f(1) = 0$

8.3 Fungsi Eksponen Asli

- Karena $D_x[\ln x] = \frac{1}{x} > 0$ untuk $x > 0$, maka fungsi logaritma asli monoton murni, sehingga mempunyai invers. Invers dari fungsi logaritma asli disebut **fungsi eksponen asli, notasi exp**. Jadi berlaku hubungan

$$y = \exp(x) \Leftrightarrow x = \ln y$$

- Dari sini didapat : $y = \exp(\ln y)$ dan $x = \ln(\exp(x))$
- **Definisi 8.2** Bilangan e adalah bilangan Real positif yang bersifat $\ln e = 1$.

Dari sifat (iv) fungsi logaritma diperoleh

$$e^r = \exp(\ln e^r) = \exp r \ln e = \exp r \quad \longrightarrow \quad \exp(x) = e^x$$

Turunan dan integral fungsi eksponen asli

Dengan menggunakan turunan fungsi invers

Dari hubungan

$$y = e^x \quad \Leftrightarrow \quad x = \ln y$$
$$\downarrow$$
$$\frac{dy}{dx} = \frac{1}{dx/dy} = y = e^x \quad \leftarrow \quad \frac{dx}{dy} = \frac{1}{y}$$

Jadi, $D_x(e^x) = e^x$

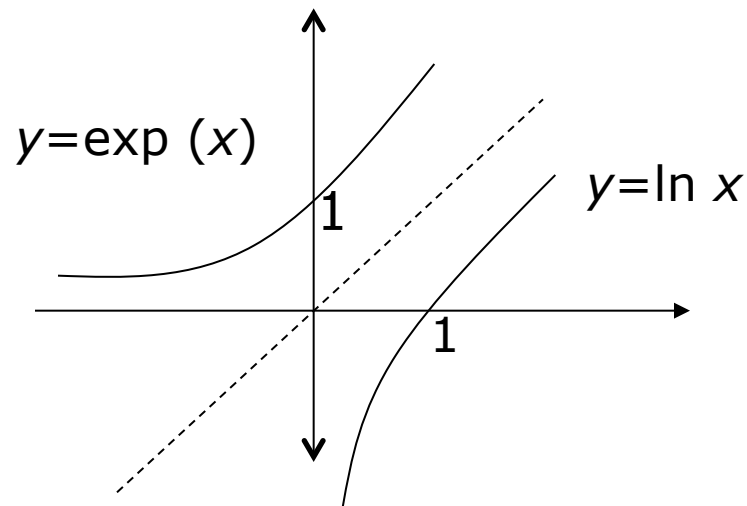
Secara umum $D_x(e^{u(x)}) = e^u \cdot u'$

Sehingga

$$\int e^x dx = e^x + C$$

Grafik fungsi eksponen asli

Karena fungsi eksponen asli merupakan invers dari fungsi logaritma asli maka grafik fungsi eksponen asli diperoleh dengan cara mencerminkan grafik fungsi logaritma asli terhadap garis $y=x$



Contoh

$$D_x(e^{3x \ln x}) = e^{3x \ln x} \cdot D_x(3x \ln x) = e^{3x \ln x} (3 \ln x + 3).$$

Contoh Hitung

$$\int \frac{e^{3/x}}{x^2} dx$$

Jawab :

Misalkan $u = \frac{3}{x} \rightarrow du = \frac{-3}{x^2} dx \rightarrow \frac{1}{x^2} dx = -\frac{1}{3} du$

Sehingga

$$\int \frac{e^{3/x}}{x^2} dx = \int -\frac{1}{3} e^u du = -\frac{1}{3} e^u + c = -\frac{1}{3} e^{3/x} + c.$$

Penggunaan fungsi logaritma dan eksponen asli

Menghitung turunan fungsi berpangkat fungsi

Diketahui $f(x) = (g(x))^{h(x)}$, $f'(x) = ?$

$$\ln(f(x)) = h(x) \ln(g(x))$$

$$D_x(\ln(f(x))) = D_x(h(x) \ln(g(x)))$$

$$\frac{f'(x)}{f(x)} = h'(x) \ln(g(x)) + \frac{h(x)}{g(x)} g'(x)$$

$$f'(x) = \left(h'(x) \ln(g(x)) + \frac{h(x)}{g(x)} g'(x) \right) f(x)$$

Contoh

Tentukan turunan fungsi $f(x) = (\sin x)^{4x}$

Jawab

Ubah bentuk fungsi pangkat fungsi menjadi perkalian fungsi dengan menggunakan fungsi logaritma asli

$$\ln f(x) = \ln(\sin x)^{4x} = 4x \ln(\sin(x))$$

Turunkan kedua ruas

$$D_x (\ln f(x)) = D_x (4x \ln(\sin(x)))$$

$$\frac{f'(x)}{f(x)} = 4 \ln(\sin(x)) + \frac{4x}{\sin x} \cos x = 4 \ln(\sin(x)) + 4x \cot x$$

$$f'(x) = (4 \ln(\sin(x)) + 4x \cot x)(\sin x)^{4x}$$

B. Selesaikan integral tak tentu berikut

1. $\int \frac{4}{2x+1} dx$

2. $\int \frac{\ln^2 3x}{x} dx$

3. $\int \frac{x^3}{x^2+1} dx$

4. $\int \frac{\tan(\ln x)}{x} dx$

5. $\int \frac{2}{x(\ln x)^2} dx$

6. $\int \frac{4x+2}{x^2+x+5} dx$

7. $\int (x+3)e^{x^2+6x} dx$

8. $\int e^{-x} \sec^2(2-e^{-x}) dx$

9. $\int (\cos x) e^{\sin x} dx$

10. $\int e^{2 \ln x} dx$

11. $\int x^2 e^{2x^3} dx$

12. $\int \frac{e^{2x}}{e^x+3} dx$

13. $\int \frac{e^{3x}}{(1-2e^{3x})^2} dx$

C. Selesaikan integral tentu berikut

$$1. \int_1^4 \frac{3}{1-2x} dx$$

$$2. \int_1^4 \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$$

$$3. \int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x + 4} dx$$

$$4. \int_0^{\ln 5} e^x(3-4e^x) dx$$

$$5. \int_0^1 e^{2x+3} dx$$

$$6. \int_0^{\ln 2} e^{-3x} dx$$

$$7. \int_1^2 \frac{e^{3/x}}{x^2} dx$$

$$8. \int_0^2 xe^{4-x^2} dx$$

$$9. \int_e^{e^2} \frac{dx}{x(\ln x)^2}$$

D. Hitung limit berikut :

1. $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$

2. $\lim_{x \rightarrow 0} (1 + \sin 2x)^{\frac{1}{x}}$

3. $\lim_{x \rightarrow \infty} \left[\cos \frac{2}{x} \right]^{x^2}$

4. $\lim_{x \rightarrow 0^+} (e^{2x} - 1)^{\frac{1}{\ln x}}$

5. $\lim_{x \rightarrow \infty} (1 + x^2)^{\frac{1}{\ln x}}$

6. $\lim_{x \rightarrow \infty} (\ln x)^{\frac{1}{x}}$

7. $\lim_{x \rightarrow \infty} (3^x + 5^x)^{\frac{1}{x}}$

8. $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x+2} \right)^x$

8.5 Fungsi Eksponen Umum

Fungsi $f(x) = a^x$, $a > 0$ disebut fungsi eksponen umum

Untuk $a > 0$ dan $x \in \mathbf{R}$, definisikan $a^x = e^{x \ln a}$

Turunan dan integral

$$D_x(a^x) = D_x(e^{x \ln a}) = e^{x \ln a} \ln a = a^x \ln a$$

Jika $u = u(x)$, maka

$$D_x(a^u) = D_x(e^{u \ln a}) = e^{u \ln a} \ln a \cdot u' = a^u u' \ln a$$

Dari sini diperoleh :

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

Sifat–sifat fungsi eksponen umum

Untuk $a > 0$, $b > 0$, x , y bilangan riil berlaku

$$1. a^x a^y = a^{x+y}$$

$$2. \frac{a^x}{a^y} = a^{x-y}$$

$$3. (a^x)^y = a^{xy}$$

$$4. (ab)^x = a^x b^x$$

$$5. \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

Contoh

1. Hitung turunan pertama dari

$$f(x) = 3^{2x+1} + 2^{\sin 2x}$$

Jawab :

$$f'(x) = 2 \cdot 3^{2x+1} \ln 3 + 2 \cdot 2^{\sin 2x} \cos 2x \ln 2$$

2. Hitung

$$\int 4^{x^2} \cdot x dx$$

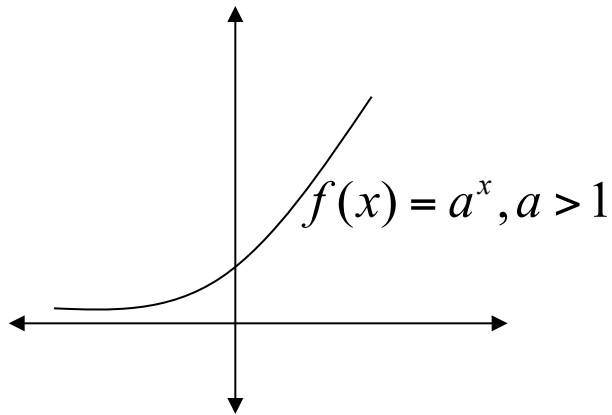
Jawab :

$$\text{Misal } u = x^2 \rightarrow du = 2x dx \rightarrow dx = \frac{1}{2x} du$$

$$\Rightarrow \int 4^{x^2} \cdot x dx = \int 4^u \frac{du}{2} = \frac{1}{2} \frac{4^u}{\ln 4} + C = \frac{4^{x^2}}{2 \ln 4} + C$$

Grafik fungsi eksponen umum

Diketahui



a. $f(x) = a^x, a > 0$

$Df = (-\infty, \infty)$

b. $f'(x) = a^x \ln a = \begin{cases} a^x \ln a < 0, & 0 < a < 1 \\ a^x \ln a > 0, & a > 1 \end{cases}$

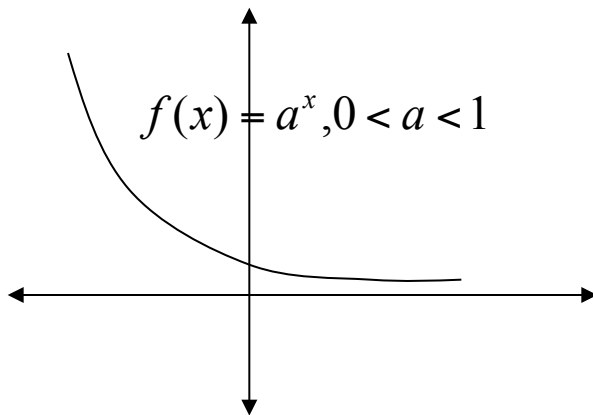
f monoton naik jika $a > 1$

monoton turun jika $0 < a < 1$

c. $f''(x) = a^x (\ln a)^2 > 0 \forall x \in D_f$

Grafik f selalu cekung keatas

d. $f(0) = 1$



8.6 Fungsi Logaritma Umum

Karena fungsi eksponen umum monoton murni maka ada Inversnya. Invers dari fungsi eksponen umum disebut fungsi Logaritma Umum (logaritma dengan bilangan pokok a), notasi ${}^a \log x$, sehingga berlaku :

$$y = {}^a \log x \Leftrightarrow x = a^y$$

Dari hubungan ini, didapat

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a} \Rightarrow {}^a \log x = \frac{\ln x}{\ln a}$$

Sehingga $D_x ({}^a \log x) = D_x \left(\frac{\ln x}{\ln a} \right) = \frac{1}{x \ln a}$

Jika $u=u(x)$, maka $D_x ({}^a \log u) = D_x \left(\frac{\ln u}{\ln a} \right) = \frac{u'}{u \ln a}$

Contoh Tentukan turunan pertama dari

$$1. f(x) = {}^3\log(x^2 + 1)$$

$$2. f(x) = {}^4\log\left(\frac{x+1}{x-1}\right)$$

Jawab :

$$1. f(x) = {}^3\log(x^2 + 1) = \frac{\ln(x^2 + 1)}{\ln 3}$$



$$f'(x) = \frac{2x}{x^2 + 1} \cdot \frac{1}{\ln 3}$$

$$2. f(x) = {}^4\log\left(\frac{x+1}{x-1}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)}{\ln 4}$$

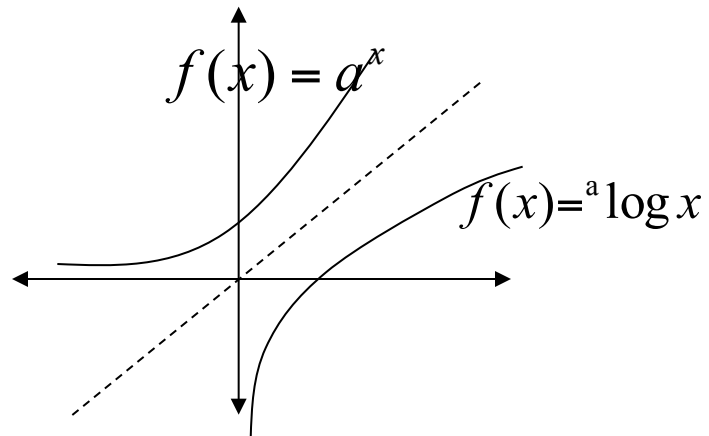


$$\begin{aligned} f'(x) &= \frac{1}{\ln 4} \cdot \frac{1}{\left(\frac{x+1}{x-1}\right)} Dx\left(\frac{x+1}{x-1}\right) \\ &= \frac{1}{\ln 4} \cdot \frac{x-1}{x+1} \cdot \frac{x-1 - (x+1)}{(x-1)^2} \\ &= \frac{1}{\ln 4} \cdot \frac{-2}{(x+1)(x-1)} \end{aligned}$$

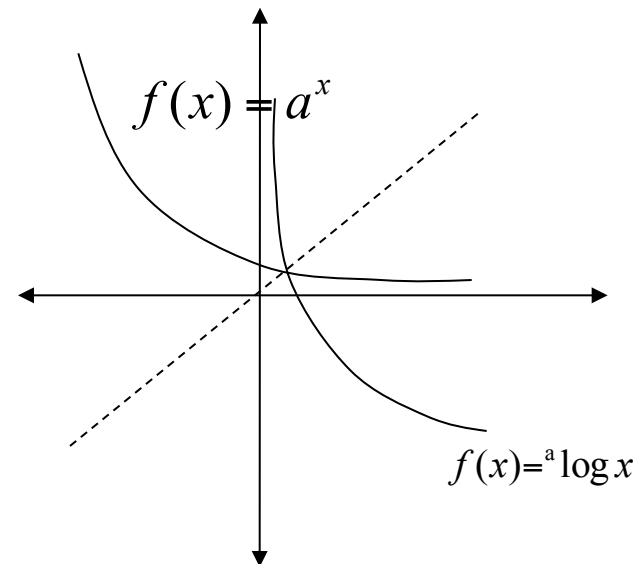
Grafik fungsi logaritma umum

Grafik fungsi logaritma umum diperoleh dengan mencerminkan grafik fungsi eksponen umum terhadap garis $y=x$

Untuk $a > 1$



Untuk $0 < a < 1$



Soal Latihan

A. Tentukan y' dari

1. $y = 3^{2x^4 - 4x}$

2. $y = {}^{10}\log(x^2 + 9)$

3. $x^3 \log(xy) + y = 2$

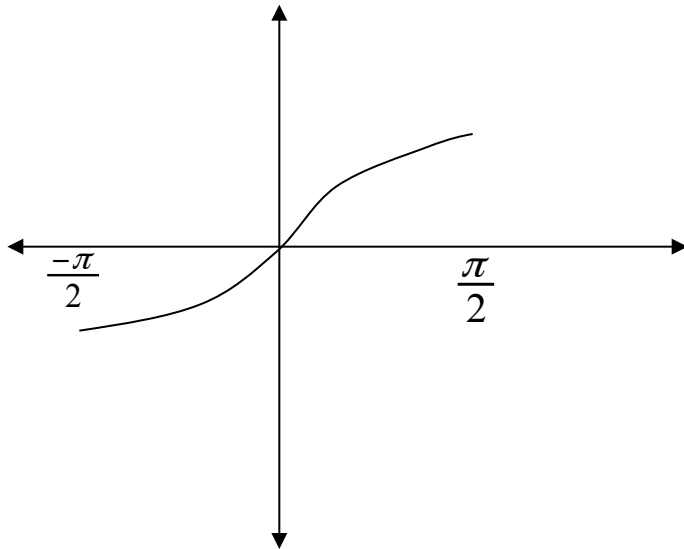
B. Hitung

1. $\int 10^{5x-1} dx$

2. $\int x 2^{x^2} dx$

8.7 Fungsi Invers Trigonometri

Fungsi trigonometri adalah fungsi yang periodik sehingga tidak satu-satu, jika daerah asalnya dibatasi, fungsi trigonometri bisa dibuat menjadi satu-satu sehingga mempunyai invers.



a. Invers fungsi sinus

Diketahui $f(x) = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Karena pada $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ $f(x) = \sin x$ monoton murni maka inversnya ada. Invers dari fungsi sinus disebut arcus sinus, notasi $\arcsin(x)$, atau $\sin^{-1}(x)$

Sehingga berlaku

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

Turunan

Dari hubungan $y = \sin^{-1} x \Leftrightarrow x = \sin y$ $-1 \leq x \leq 1, \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
dan rumus turunan fungsi invers diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}, |x| < 1$$

atau $D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

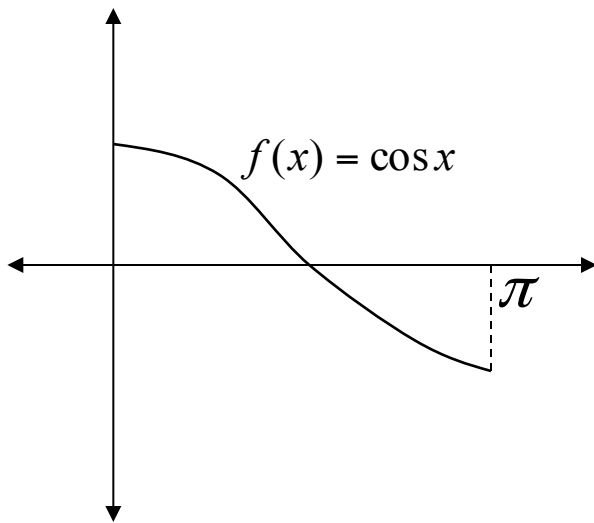
Jika $u=u(x)$ $D_x(\sin^{-1} u) = \frac{u'}{\sqrt{1-u^2}}$

Dari rumus turunan diperoleh

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

b. Invers fungsi cosinus

Fungsi $f(x) = \cos x$, $0 \leq x \leq \pi$ monoton murni (selalu monoton turun), sehingga mempunyai invers



Definisi : Invers fungsi $\cos x$ disebut arcuscos x , notasi $\arccos x$ atau $\cos^{-1}(x)$

Berlaku hubungan

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

Turunan

Dari $y = \cos^{-1} x \Leftrightarrow x = \cos y$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$ diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - \cos^2 y}} = \frac{-1}{\sqrt{1 - x^2}}, |x| < 1$$

atau $D_x(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

Jika $u = u(x)$ $D_x(\cos^{-1} u) = \frac{-u'}{\sqrt{1-u^2}}$

Dari rumus turunan diatas diperoleh

$$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$$

Contoh

$$D_x(\sin^{-1}(x^2)) = \frac{1}{\sqrt{1-(x^2)^2}} D_x(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

$$D_x(\cos^{-1}(\tan x)) = \frac{-1}{\sqrt{1-(\tan x)^2}} D_x(\tan x) = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$$

Contoh Hitung

$$\int \frac{1}{\sqrt{4-x^2}} dx$$

Jawab :

Gunakan rumus

$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) + C$$

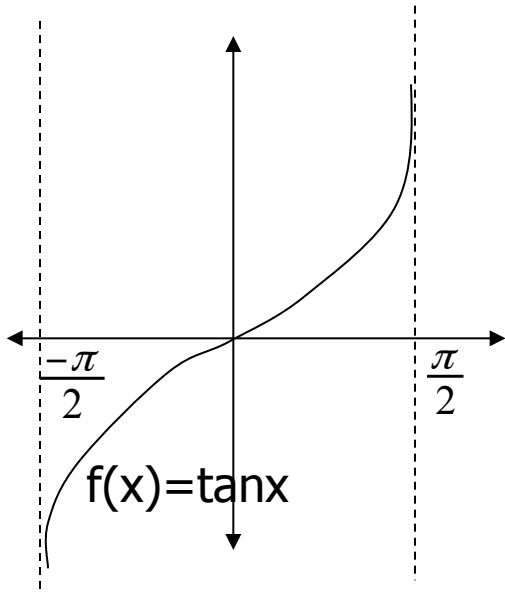
$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{4\left(1-\frac{x^2}{4}\right)}} dx = \frac{1}{2} \int \frac{1}{\sqrt{\left(1-\left(\frac{x}{2}\right)^2\right)}} dx$$

Misal $u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$

$$\int \frac{1}{\sqrt{4-x^2}} dx = \frac{1}{2} \int \frac{2}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1} \left(\frac{x}{2}\right) + C$$

c. Invers fungsi tangen

Fungsi $f(x) = \tan x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ Monoton murni (selalu naik) sehingga mempunyai invers.



Definisi Invers dari $\tan x$ disebut fungsi arcus tangen, notasi arc tangen atau $\tan^{-1}(x)$

Berlaku hubungan

$$y = \tan^{-1} x \Leftrightarrow x = \tan y$$

Turunan

Dari $y = \tan^{-1} x \Leftrightarrow x = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ dan turunan fungsi invers diperoleh

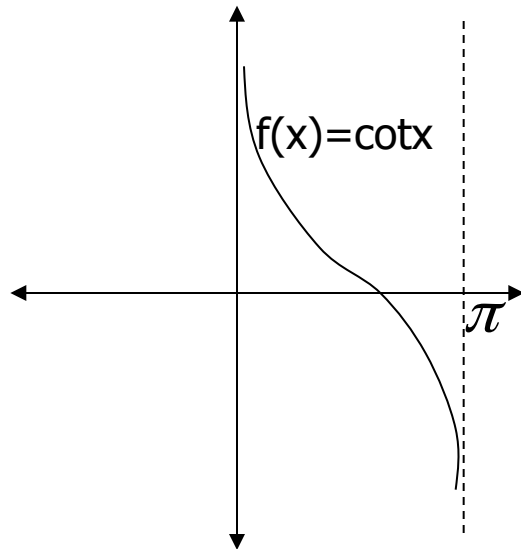
$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

atau $D_x(\tan^{-1} x) = \frac{1}{1+x^2} \longrightarrow \int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Jika $u=u(x)$ $D_x(\tan^{-1} u) = \frac{u'}{1+u^2}$

d. Invers fungsi cotangen

Fungsi $f(x) = \cot x$, $0 < x < \pi$ selalu monoton turun(monoton murni) sehingga mempunyai invers



Definisi Invers dari fungsi $\cot x$ disebut Arcus $\cot x$, notasi $\text{arc } \cot x$ atau $\cot^{-1} x$

Berlaku hubungan

$$y = \cot^{-1} x \Leftrightarrow x = \cot y$$

Turunan

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{\csc^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

atau $D_x(\cot^{-1} x) = \frac{-1}{1+x^2} \longrightarrow \int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

Jika $u=u(x)$ $D_x(\cot^{-1} u) = \frac{-u'}{1+u^2}$

Contoh

$$D_x(\tan^{-1}(x^2 + 1)) = \frac{1}{1+(x^2 + 1)^2} Dx(x^2 + 1) = \frac{2x}{1+(x^2 + 1)^2}$$

$$D_x(\cot^{-1}(\sin x)) = \frac{-1}{1+(\sin x)^2} Dx(\sin x) = \frac{-\cos x}{1+\sin^2 x}$$

Contoh Hitung

a. $\int \frac{dx}{4+x^2}$

b. $\int \frac{dx}{x^2 + 2x + 4}$

Jawab

$$\text{a. } \int \frac{1}{4+x^2} dx = \int \frac{1}{4\left(1+\frac{x^2}{4}\right)} dx = \frac{1}{4} \int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx$$

$$u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$$

$$\begin{aligned} \int \frac{1}{4+x^2} dx &= \frac{1}{4} \int \frac{2}{1+u^2} du = \frac{1}{2} \tan^{-1} u + C \\ &= \frac{1}{2} \tan^{-1} \left(\frac{x}{2}\right) + C \end{aligned}$$

Gunakan rumus

$$\int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$$

$$\begin{aligned}
 \text{b. } \int \frac{dx}{x^2 + 2x + 4} &= \int \frac{1}{(x+1)^2 + 3} dx = \int \frac{1}{3\left(1 + \frac{(x+1)^2}{3}\right)} dx \\
 &= \frac{1}{3} \int \frac{1}{1 + \left(\frac{(x+1)}{\sqrt{3}}\right)^2} dx
 \end{aligned}$$

Misal $u = \frac{x+1}{\sqrt{3}} \rightarrow du = \frac{1}{\sqrt{3}} dx \rightarrow dx = \sqrt{3} du$

Gunakan rumus

$$\int \frac{1}{1+u^2} du = \tan^{-1}(u) + C$$

$$\begin{aligned}
 \int \frac{dx}{x^2 + 2x + 4} &= \frac{1}{3} \int \frac{\sqrt{3}}{1+u^2} du = \frac{1}{\sqrt{3}} \tan^{-1} u + C \\
 &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) + C
 \end{aligned}$$

e. Invers fungsi secan

Diberikan $f(x) = \sec x$, $0 \leq x \leq \pi$, $x \neq \frac{\pi}{2}$

$$f'(x) = \sec x \tan x > 0, 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$$



$f(x) = \sec x$ monoton murni



Ada inversnya

Definisi Invers dari fungsi $\sec x$ disebut arcus secx, notasi $\text{arc sec } x$ atau $\sec^{-1} x$

Sehingga

$$y = \sec^{-1} x \Leftrightarrow x = \sec y$$

Turunan

Dari $y = \sec^{-1} x \Leftrightarrow x = \sec y$

$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

$$\begin{array}{ccc} \downarrow & & \uparrow \\ \cos y = \frac{1}{x} & \longrightarrow & y = \cos^{-1}\left(\frac{1}{x}\right) \end{array}$$

Sehingga

$$D_x(\sec^{-1} x) = D_x\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \cdot \frac{-1}{x^2} = \frac{|x|}{x^2 \sqrt{x^2 - 1}} = \frac{1}{|x| \sqrt{x^2 - 1}}$$

Jika $u = u(x)$ $D_x(\sec^{-1} u) = \frac{u'}{|u| \sqrt{u^2 - 1}}$

$$\longrightarrow \int \frac{1}{x \sqrt{x^2 - 1}} dx = \sec^{-1} |x| + c$$

e. Invers fungsi cosecan

Diberikan $f(x) = \csc x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $x \neq 0$

$$f'(x) = -\csc x \cot x < 0, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$$



$f(x) = \csc x$ monoton murni



Ada inversnya

Definisi Invers dari fungsi $\csc x$ disebut arcus $\csc x$, notasi $\text{arc csc } x$ atau $\csc^{-1} x$

Sehingga

$$y = \csc^{-1} x \Leftrightarrow x = \csc y$$

Turunan

Dari $y = \csc^{-1} x \Leftrightarrow x = \csc y$

$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

$$\begin{array}{ccc} \downarrow & & \uparrow \\ \sin y = \frac{1}{x} & \longrightarrow & y = \sin^{-1}\left(\frac{1}{x}\right) \end{array}$$

Sehingga

$$D_x(\csc^{-1} x) = D_x\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \cdot \frac{-1}{x^2} = \frac{-|x|}{x^2 \sqrt{x^2 - 1}} = \frac{-1}{|x| \sqrt{x^2 - 1}}$$

Jika $u = u(x)$ $D_x(\sec^{-1} u) = \frac{-u'}{|u| \sqrt{u^2 - 1}}$

$$\longrightarrow \int \frac{1}{x \sqrt{x^2 - 1}} dx = -\csc^{-1} |x| + c$$

Contoh

A. Hitung turunan pertama dari

a. $f(x) = \sec^{-1}(x^2)$

b. $f(x) = \sec^{-1}(\tan x)$

Jawab

a.
$$f'(x) = \frac{1}{|x^2| \sqrt{(x^2)^2 - 1}} Dx(x^2) = \frac{2x}{x^2 \sqrt{x^4 - 1}} = \frac{2}{x \sqrt{x^4 - 1}}$$

b.
$$f'(x) = \frac{1}{|\tan x| \sqrt{(\tan x)^2 - 1}} Dx(\tan x) = \frac{\sec^2 x}{|\tan x| \sqrt{\tan^2 x - 1}}$$

B. Hitung

$$\int \frac{1}{x\sqrt{x^2 - 4}} dx$$

Jawab

$$\int \frac{1}{x\sqrt{x^2 - 4}} dx = \int \frac{1}{x\sqrt{4\left(\frac{x^2}{4} - 1\right)}} dx = \frac{1}{2} \int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2 - 1}} dx$$

$$\text{Misal } u = \frac{x}{2} \rightarrow du = \frac{1}{2} dx \rightarrow dx = 2du$$

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2 - 4}} dx &= \frac{1}{2} \int \frac{1}{2u\sqrt{u^2 - 1}} 2du = \frac{1}{2} \int \frac{1}{u\sqrt{u^2 - 1}} du \\ &= \frac{1}{2} \sec^{-1} |u| + C = \frac{1}{2} \sec^{-1} \left| \frac{x}{2} \right| + C \end{aligned}$$

Soal Latihan

A. Tentukan turunan pertama fungsi berikut, sederhanakan jika mungkin

1. $y = (\sin^{-1} x)^2$

2. $y = \tan^{-1}(e^x)$

3. $y = \tan^{-1} x \ln x$

4. $f(t) = e^{\sec^{-1} t}$

5. $y = x^2 \cot^{-1}(3x)$

6. $y = \tan^{-1}(x - \sqrt{1 + x^2})$

B. Hitung

$$1. \int \frac{dx}{9x^2 + 16}$$

$$2. \int \frac{dx}{4x\sqrt{x^2 - 16}}$$

$$3. \int \frac{dx}{\sqrt{2 - 5x^2}}$$

$$4. \int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$$

$$5. \int \frac{e^x}{e^{2x} + 1} dx$$

$$6. \int \frac{e^{2x}}{\sqrt{1 - e^{4x}}} dx$$

$$7. \int \frac{dx}{x[4 + (\ln x)^2]}$$

8.8 Fungsi Hiperbolik

Definisi

- a. Fungsi kosinus hiperbolik : $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$
- b. Fungsi sinus hiperbolik : $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$
- c. Fungsi tangen hiperbolik : $f(x) = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- d. Fungsi cotangen hiperbolik : $f(x) = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
- e. Fungsi secan hiperbolik : $f(x) = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
- f. Fungsi cosecan hiperbolik : $f(x) = \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Persamaan identitas pada fungsi hiperbolik

$$\left. \begin{array}{l} 1. \cosh x + \sinh x = e^x \\ 2. \cosh x - \sinh x = e^{-x} \end{array} \right\} 3. \cosh^2 x - \sinh^2 x = 1$$
$$4. 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$5. \operatorname{coth}^2 x - 1 = \operatorname{cosech}^2 x$$

Turunan

$$D_x(\cosh x) = D_x\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \quad \Rightarrow \quad \int \sinh x \, dx = \cosh x + C$$

$$D_x(\sinh x) = D_x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \quad \Rightarrow \quad \int \cosh x \, dx = \sinh x + C$$

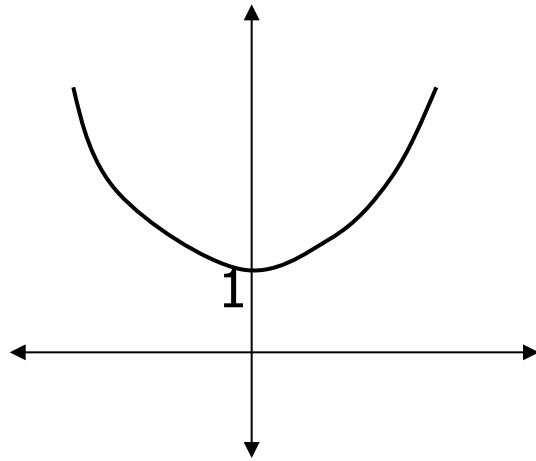
$$D_x(\tanh x) = D_x\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\begin{aligned} D_x(\operatorname{coth} x) &= D_x\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x \end{aligned}$$

$$D_x(\operatorname{sech} hx) = D_x\left(\frac{1}{\cosh x}\right) = \frac{-\sinh x}{\cosh^2 x} = -\operatorname{sech} hx \tanh x$$

$$D_x(\operatorname{cosech} hx) = D_x\left(\frac{1}{\sinh x}\right) = \frac{-\cosh x}{\sinh^2 x} = -\operatorname{cosech} hx \operatorname{coth} x$$

Grafik $f(x) = \cosh x$



Diketahui

$$(i) \quad f(x) = \cosh x = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

$$(ii) \quad f'(x) = \frac{e^x - e^{-x}}{2} = \begin{cases} f'(x) < 0, & x < 0 \\ f'(x) > 0, & x > 0 \end{cases}$$

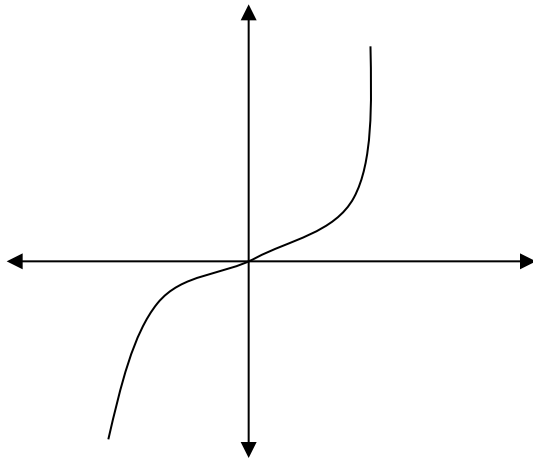
f monoton naik pada $x > 0$
 monoton turun pada $x < 0$

$$(iii) \quad f''(x) = \frac{e^x + e^{-x}}{2} > 0, \forall x \in \mathbb{R}$$

Grafik f selalu cekung keatas

$$(iv) \quad f(0) = 1$$

Grafik $f(x) = \sinh x$



Diketahui

$$(i) \quad f(x) = \sinh x = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

$$(ii) \quad f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

f selalu monoton naik

$$(iii) \quad f''(x) = \frac{e^x - e^{-x}}{2} = \begin{cases} > 0, x > 0 \\ < 0, x < 0 \end{cases}$$

Grafik f cekung keatas pada $x > 0$
cekung kebawah pada $x < 0$

$$(iv) \quad f(0) = 0$$

Contoh

Tentukan y' dari

1. $y = \tanh(x^2 + 1)$

2. $x^2 \sinh x + y^2 = 8$

Jawab

1. $y' = \operatorname{sech}^2(x^2 + 1) D_x(x^2 + 1) = 2x \operatorname{sech}^2(x^2 + 1)$

2. $D_x(x^2 \sinh x + y^2) = D_x(8)$

$$2x \sinh x + x^2 \cosh x + 2y y' = 0$$

$$y' = -\frac{2x \sinh x + x^2 \cosh x}{2y}$$

Soal Latihan

A. Tentukan turunan pertama dari

1. $f(x) = \tanh 4x$

2. $g(x) = \sinh^2 x$

3. $g(x) = \frac{1 - \cosh x}{1 + \cosh x}$

4. $h(t) = \coth \sqrt{1 + t^2}$

5. $g(t) = \ln(\sinh t)$

6. $f(x) = x \cosh x^2$

B. Hitung integral berikut

1. $\int \text{Sinh}(1 + 4x) dx$

2. $\int \sinh x \cosh^2 x dx$

3. $\int \tanh x dx$

4. $\int \frac{\text{sech}^2 x}{2 + \tanh x} dx$